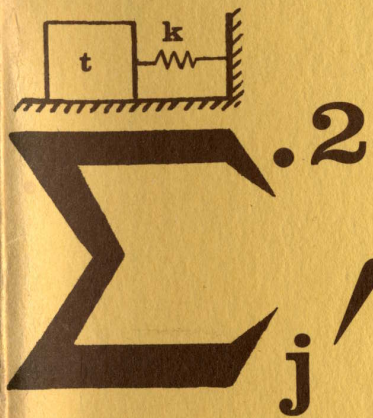
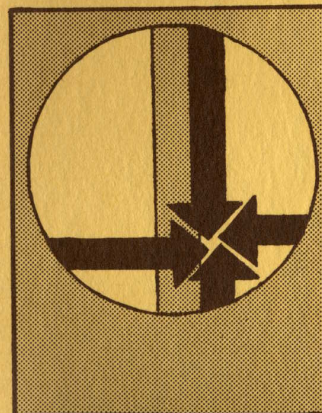




mechanics series

Monograph M-1



**American Society for
Engineering Education**

**mechanics
series**

Monograph M-1

MECHANICS DIVISION OFFICERS

Chairman—Philip J. Erdle, United States Air Force Academy
Program Chairman—Steven C. Batterman, Pennsylvania State University
Secretary—Douglas C. Dowell, United States Air Force Academy
Editor—Joseph C. Osborn, Lehigh University
Executive Committee—
Steven C. Batterman, University of Pennsylvania
Edward F. Byars, West Virginia University
Paul C. Jennings, California Institute of Technology
E. Russell Johnston, Jr., University of Connecticut
Milton E. Raville, Georgia Institute of Technology
Grover L. Rogers, Florida State University
Richard Skalak, Columbia University

© 1969 by the American Society for Engineering Education
Library of Congress Catalog Card No. 72-93552

Order From:

American Society for Engineering Education
Suite 838, 2100 Pennsylvania Avenue, N. W.
Washington, D. C. 20037

Price:

\$2.00 (Orders must be prepaid unless a purchase order is submitted.)

CONTENTS

iv	Dedication to Stephen P. Timoshenko
1	Compressed Air-Water Rocket for Deformable Control Volume Analyses J. F. Foss—Associate Professor of Mechanical Engineering, Michigan State University D. E. Heyer—Engineering Development Laboratory, E.I. du Pont de Nemours and Company
10	Elementary Theory of Axially Loaded Springs Morris Ojalvo—Professor of Civil Engineering, Ohio State University John T. Wilson—Graduate Student, Ohio State University
14	Growth of the Turbulent Flat-Plate Boundary Layer James M. Robertson—Professor of Theoretical and Applied Mechanics, University of Illinois
22	Experimental Mechanics for Undergraduates Bruce G. Rogers—Professor of Civil Engineering, Lamar State College of Technology
26	St. Venant's Problem in Cylindrical Coordinates Robert Schmidt—Professor of Engineering Sciences, University of Detroit

DEDICATION



Stephen P. Timoshenko
Scientist, Engineer, Teacher
1878-

The appearance of this first Mechanics Division Monograph coincides closely with the 90th birthday of Professor Emeritus Stephen P. Timoshenko of Stanford University. In view of Professor Timoshenko's lifelong dedication to the advancement of mechanics, it is fitting that this monograph should be dedicated to him on such an occasion.

Throughout his long and active career, which extends from Czarist Russia across Europe and finally to America, Professor Timoshenko has held steadfastly to one goal: to further the advancement of mechanics as a science and to promote its application to practical engineering problems. The events of his interesting and exciting life are fully recorded in his autobiography, *As I Remember* (Van Nostrand, 1968), and need not be repeated here.

In Europe he acquired the sound scientific and mathematical training that enabled him to bring the full power of his talent to bear upon new problems. In Germany he came under the influence of Felix Klein, who inspired him with the importance of bringing theory closer to application. Then, in America, he was forced by circumstances to abandon temporarily his dream of teaching. Instead, he spent five years at Westinghouse Corporation, where he learned to know intimately the real-life problems of engineering at a time when industry in the United States was first beginning to feel the need for more theoretical knowledge.

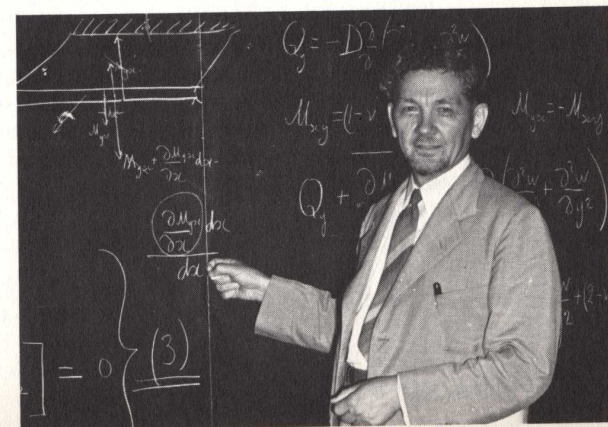
At the age of 49, Timoshenko left industry to go to the University of Michigan, where he began a most remarkable teaching career. His

fame spread rapidly, and soon he was surrounded by graduate students from all parts of America and abroad. Those who attended his lectures marveled at the ease and clarity with which he explained the most difficult subjects. He always chose the simplest and most direct explanation. He was so much in tune with his listeners that he would sense a question and answer it before it could be asked—certainly a sign of his genius as a teacher.

For almost three decades—at Westinghouse Corporation, the University of Michigan, and Stanford University—Professor Timoshenko promoted mechanics both in and out of the university. There was such increase in the subject's prestige during these years that they are rightly referred to as the Timoshenko Era in applied mechanics. Those who witnessed his profound influence on American engineering education, coupled with his reluctance to adapt himself to American ways, aptly remarked that he had succeeded in producing more change in America than America had produced in him.

In his autobiography Professor Timoshenko discusses his deliberations on whether or not to remain in the United States after his first visit here in 1922. After describing the advantages and disadvantages of the matter, he said, "After long hesitation I decided to stay in America. Whether I chose rightly or wrongly, I do not know even now, after some forty years." May we say to Professor Timoshenko that we believe he chose rightly, and we are glad that he chose as he did.

—D. H. YOUNG



FOREWORD

This monograph, which is the first of an annual series, replaces the Bulletin of the Mechanics Division. The Bulletin first appeared in fall 1950, with Dr. Glenn Murphy of Iowa State University as the editor. He had been the Division Chairman for 1949-50 and was instrumental in initiating the publication. From fall 1950 to spring 1955, the Bulletin was a separate entity, published with the cooperation and support of The Macmillan Company. Dr. Murphy was editor for three years; he was then succeeded by Dr. Arthur W. Davis, also of Iowa State. Dr. Davis agreed to take the editorship temporarily, but he continued for ten years, from fall 1953 until spring 1963.

In spring 1955, The Macmillan Company decided that it could no longer support the Bulletin. In spite of efforts by the editor and Executive Committee of the Mechanics Division, no other financial support was obtained. Hence, there was a lull from May 1955 until December 1956, when the Bulletin was resumed as a part of the *Journal of Engineering Education* upon the invitation of its editor, E.C. McClintock, Jr. This arrangement continued until February 1968.

The first year (1950-51) there were four issues of the Bulletin. Volume I Number 1 was a mimeographed publication that was primarily a newsletter to the Division membership, but it did contain one article on mechanics teaching by Dr. Arthur Davis. The other three issues were printed in an attractive pamphlet. From 1950 to 1958, most issues of the Bulletin contained a cartoon by Dr. Elmer L. Munger depicting the classroom performance of Prof. Z. Freebody Snafu. In June 1963, Dr. Davis asked to be relieved as editor, and he was succeeded by the present editor, Dr. Joseph C. Osborn of Lehigh University.

With the appearance of this monograph, publication of articles of primary interest to the members of the Mechanics Division enters a new phase. The present plan is to have a series of annual monographs, each volume appearing in the fall. The newsletter is to be revived, probably appearing in the spring and containing such items as the Chairman's message and announcements concerning Division activities, particularly those planned for the ASEE Annual Meeting.

Comments about this monograph may be sent either to the Division editor or to any of the Division officers.

Compressed Air-Water Rocket for Deformable Control Volume Analyses

J. F. FOSS
D. E. HEYER

An analysis of the compressed air-water rocket used the deformable control volume form of the continuity, momentum, and energy equations. The resulting simultaneous, first-order, nonlinear differential equations for the water efflux and rocket velocities have been evaluated by numerical solution. High-speed photography was used to record and analyze the physical event. Satisfactory agreement between experimental and analytical values is obtained.

The control volume formulation of the conservation laws—mass, momentum, and energy—permits the engineer to deal with complex problems on a deductive basis. That is, since the generalized equations include all the pertinent effects, a term-by-term analysis will isolate the need for appropriate and necessary assumptions and restrictions. Since developing a broadly based problem-solving capability is a primary objective of undergraduate engineering education, it is not surprising that nearly all recent fluid mechanics textbooks include a generalized control volume approach to the conservation laws. Also, the common technique of deriving the form of the equation relating the systems statement to the control volume statement for any extensive property deepens student understanding of the fundamental elements of the control volume formulation. The importance of providing the student with a firm understanding of the control volume formulation is accentuated because he is not likely to be exposed to it again in graduate work or professional training programs.

Three laboratory experiments which have proved valuable in the overall instruction pattern relating to the control volume have been studied (Foss, 1968). This paper, an abridgment of the study, describes the analytical-numerical-experimental elements of the compressed air-water rocket problem and indicates its use in the undergraduate course. The complete report is available (Foss and Heyer).

The compressed air-water rocket (Fig. 1) is a commercially available toy. As a toy, it operates by filling the shell with water to a prescribed level, pressurizing the air above the liquid with a hand pump, and obtaining flight as the pressurized air ejects the water upon release of the holding fingers. Fig. 1 shows a modified system used for the experimental evaluation of the rocket flight. The problem under consideration is to describe the rocket dynamics—acceleration, velocity, position—as a function of time for arbitrary initial conditions—water mass, air pressure.

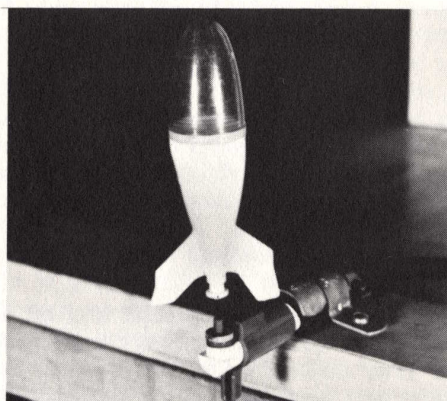


Fig. 1. Experimental equipment, launcher and rocket.

The important instructional features of the problem may be extracted to better indicate its pedagogic value:

1. An accelerating-elevating physical device which requires special consideration for the reference frames from which to evaluate the kinetic and potential energy terms and the flux and time rate of change terms of the momentum equation.
2. Thermodynamic considerations, e.g., an isentropic expansion of the pressurized gas.
3. The problem's inherent time dependence, involving the unsteady flow terms.
4. Interdependence of rocket and water motions, necessitating a simultaneous solution of continuity, momentum, and energy equations.
5. The difficulty in solving the problem without the organizational and computational contributions of the generalized deformable control volume formulation.

ANALYTICAL MODEL AND COMPUTATIONAL TECHNIQUE

The noninertial, deformable control volume forms of the energy, momentum, and continuity equations were used for the solution of the compressed air-water rocket problem. The equations take the following forms for rectilinear motion (Hansen).

Energy:

$$\dot{Q} = \dot{W}_s + \frac{d}{dt} \int_{V_{c.v.}} \rho \left[u + \frac{V^2}{2} + gz \right] dv + \int_{S_{c.v.}} \rho \left[u + \frac{V^2}{2} + gz + p/\rho \right] V_{rn} dA + \int_{S_{c.v.}} p \vec{V}_b \cdot \hat{n} dA \quad (1)$$

Momentum (z-component):

$$\hat{k} \cdot [(\vec{g} - \vec{a}_O) \int_{V_{c.v.}} \rho dv + \int_{S_{c.v.}} \vec{\tau} dA - \int_{S_{c.v.}} p \hat{n} dA + \vec{F}_{mechanical}] = \hat{k} \cdot \left[\frac{d}{dt} \int_{V_{c.v.}} \rho \vec{V}_r dv + \int_{S_{c.v.}} \rho \vec{V}_r V_{rn} dA \right] \quad (2)$$

Continuity:

$$0 = \frac{d}{dt} \int_{V_{c.v.}} \rho dv + \int_{S_{c.v.}} \rho V_{rn} dA \quad (3)$$

The energy equation as stated above is valid for a noninertial reference frame only if the terms are properly evaluated. Considering a reference frame attached to the earth to be inertial, a velocity relative to the inertial frame is expressed as

$$\vec{V} = \vec{V}_O + \vec{V}_r;$$

thus, it follows that

$$V^2 = V_O^2 + 2\vec{V}_O \cdot \vec{V}_r + V_r^2.$$

Similarly,

$$z = z_O + z_r.$$

The application of these results to Eq. 1 yields

$$\begin{aligned} \dot{Q} = & \dot{W}_s + \frac{d}{dt} \int_{V_{c.v.}} \rho \left[u + \frac{V_r^2}{2} + \vec{V}_O \cdot \vec{V}_r + gz_r \right] dv \\ & + \frac{d}{dt} \int_{V_{c.v.}} \rho \left[\frac{V_O^2}{2} + gz_O \right] dv + \int_{S_{c.v.}} \rho \left[u + p/\rho + \frac{V_O^2 + V_r^2}{2} \right. \\ & \left. + \vec{V}_O \cdot \vec{V}_r + g(z_O + z_r) \right] V_{rn} dA + \int_{S_{c.v.}} p \vec{V}_b \cdot \hat{n} dA. \end{aligned} \quad (4)$$

Eq. 2, 3, and 4 constitute the generalized forms of the momentum, continuity, and energy equations. To apply these to the compressed air-water

rocket problem, it is necessary to select an appropriate control volume and to introduce appropriate restrictions and assumptions which will allow the computing equations to be developed. The choice of control volume was influenced by the desire to include all pertinent effects in as straightforward a manner as possible and to simplify development of the computing equations by dealing with only one control volume. The control volume shown in Fig. 2 was selected because it includes the external surface forces in the momentum and energy equations as drag forces and rate-of-work terms respectively; the time variable water mass is conveniently accounted for by the deformable control surface at the air-water interface; there is no influx to the control volume and the efflux is readily accounted for at the exit plane of the rocket nozzle; and there are no mechanical forces to consider in the momentum equation.

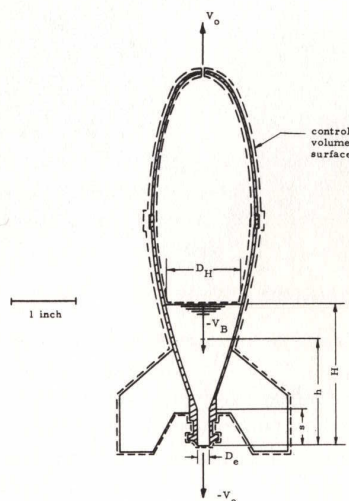


Fig. 2. Compressed air-water rocket, control volume, and nomenclature.

The following assumptions and restrictions are introduced: the flow is one-dimensional, i.e., viscous effects are assumed to be negligible; the rocket is assumed to follow a vertical path; air is assumed to be a perfect gas, and the expansion process of the air in the rocket is considered isentropic; and the temperature of the water is assumed constant throughout the control volume and invariant with respect to time.

The analytical model is based on the interval from release of the holding mechanism (Fig. 1) to the condition for which water is completely ejected from the rocket. Details of the initial lift-off, where the rocket rises past the short (3/16 in.) guide tube, are not considered; instead, the model is employed for the entire process.

An evaluation of each term in Eq. 2, 3, and 4 has been made using the restrictions noted above; these detailed considerations are presented in the complete

study (Foss and Heyer). The continuity equation may be combined with the reduced momentum and energy equations to establish two nonlinear, first-order time-dependent, ordinary differential equations. The computing equations are

Energy:

$$A \dot{V}_e + B \dot{V}_o + C = 0; \quad (5)$$

Momentum:

$$D \dot{V}_e + E \dot{V}_o + F = 0, \quad (6)$$

where

$$A = (s + H \frac{V_o}{V_e} + D_e^2 \int_s^H \frac{dh}{D^2}) \rho A_e V_e$$

$$B = \rho H A_e V_e + V_o (M_W + M_R)$$

$$C = [\frac{V_e^2}{2} (\frac{D_e}{D_H})^4 + V_o V_e (\frac{D_e}{D_H})^2 + gH + \frac{p_a \cdot p_e}{\rho} - \frac{V_e}{2} (2V_o + V_e)] \rho A_e V_e + V_o (M_W + M_R) g + \frac{1}{2} C_D \rho V_o^3 A$$

$$D = \rho A_e H$$

$$E = M_W + M_R$$

$$F = V_e [(\frac{D_e}{D_H})^2 - 1] \rho A_e V_e + (M_W + M_R) g + \frac{1}{2} C_D \rho V_o^2 A$$

From these coefficients it is clear that the equations are nonlinear and that a general analytical solution would be quite difficult to obtain. Fortunately, the equations do allow a rather direct application of a forward-stepping numerical solution. For this purpose, Eq. 5 and 6 are treated as algebraic equations for $\dot{V}_o$ and $\dot{V}_e$. The values of the coefficients A through F can be determined for known values of V_o , V_e , H , D_H , M_W , p_a , and C_D . The initial conditions are $V_o(0) = 0$ and, from the Bernoulli equation (Foss and Heyer),

$$V_e(0) = - \left[\frac{2}{(1 - A_e^2/A_H^2)} \frac{p_a - p_{atm}}{\rho} \right]^{1/2}$$

Other considerations relate the time-dependent water mass to the initial water mass and the time integral of the exiting water mass. The isentropic relationship

$$p_a(t) = p_{aI} \left[\frac{v_{aI}}{v_{aI} - A_e \int_0^t V_e dt} \right]^k$$

is used to specify the time-dependent air pressure above the water.

EXPERIMENTAL EQUIPMENT AND PROCEDURE

Fig. 3 is a schematic drawing of the pressurization system which makes use of shop air, a pressure regulator, and a pressure gauge. The locking fingers and the ball check valve of the original hand-operated launcher were retained for the tests.

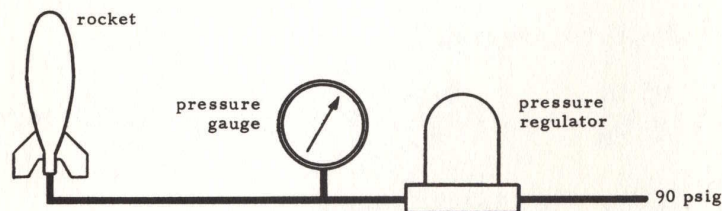


Fig. 3. Experimental equipment, pressurization system.

The numerical solution of the equations developed in the analytical model indicated that the water would be completely ejected within 0.07 sec. at most, and that maximum elevation attained by the rocket in that time would be 1.5 ft. A Fastex high-speed motion picture camera was used to determine the elevation-time histories of the rocket during this interval. A nominal film speed of 1800 frames per sec. was used for four initial air pressure and water mass conditions. The film from which the experimental data were obtained is available on loan or at cost (Foss and Heyer).

Data reduction was accomplished with a projector which allowed frame-by-frame advancement of the film. A reference marker for the elevation allowed measurements on the screen to be conveniently converted to real inches. A timing light exposed the film's edge at a frequency of 120 cycles per sec., allowing an accurate determination of the film speed.

RESULTS, CONCLUSIONS, AND INSTRUCTIONAL USE

A comparison of the experimental and numerical values for the rocket height as a function of time is presented in Fig. 4. The details by which these data were generated and additional results may be found in Eq. 2; however, this comparison is quite adequate to indicate the satisfactory nature of the model.

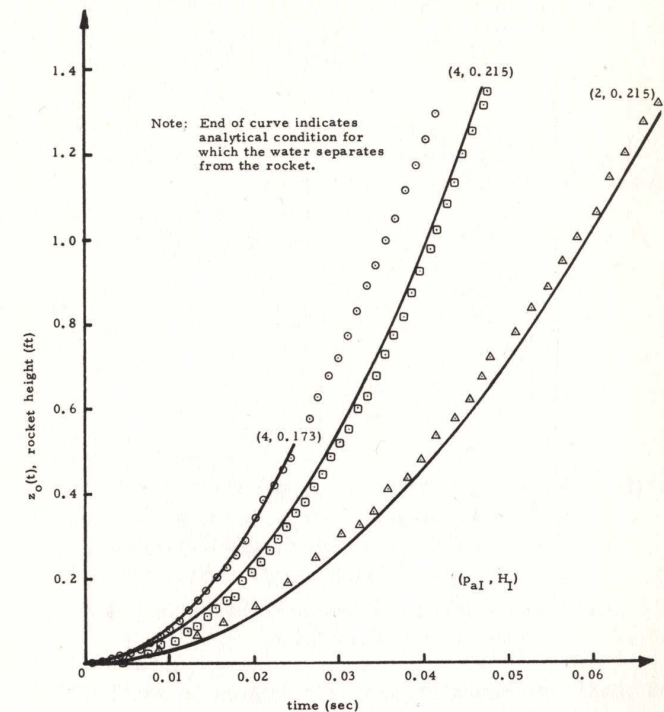


Fig. 4. Numerical and experimental results for $z_0(t)$, three initial conditions.

A final point regarding the use of this problem to exemplify the capabilities provided by the control volume formulation includes the following considerations. In the limiting cases of having a very small mass of water or having the initial water volume nearly equal to the total volume of the rocket, it is reasonable to expect that an optimum water mass exists for a given initial value of air pressure in order to obtain maximum rocket height. If the additional thrust due to the compressed air remaining in the rocket after separation is neglected, and the rocket elevation at separation is considered negligible with respect to final maximum height attained by the rocket, then a condition to optimize the final height of the rocket would be to maximize V_0 at separation. Fig. 5 presents the analytical results for V_0 at separation as a function of the initial water mass. The optimum initial water mass is indicated by the maxima exhibited by the curves of Fig. 5.

Thus, from the analytical considerations, the engineering question of the optimum water mass may be answered, and from the analysis of Eq. 2, considerations of an optimum rocket design may be formulated.

To date, use of this problem has been limited to a classroom presentation. The analysis is developed with descriptive indications of special considerations, such as the necessity of referencing kinetic energy changes to a fixed reference

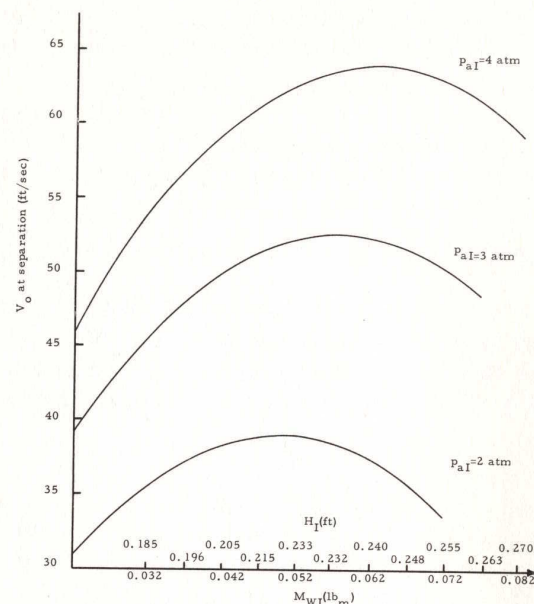


Fig. 5. Numerical results to determine optimum water mass for maximum V_o at separation.

frame, and results are presented. Emphasis is given to the fact that a fluid mechanics analysis allows an engineering consideration—optimization—to be made. Efforts are presently under way to incorporate relevant elements of this study so as to involve the student in the application of basic equations to this physical problem.

NOMENCLATURE

a	Acceleration.
A	Area. Without subscript, "A" refers to a characteristic area of the rocket used to express the drag force. Note subscript nomenclature for other designations.
C_D	Drag coefficient for rocket (assumed to be 0.2).
D	Internal diameter of rocket (Fig. 1). Note subscript designations.
g	Acceleration due to gravity.
h	Height above plane of rocket nozzle exit (Fig. 1).
H	Height at water surface (Fig. 1). $H = H(t)$.
k	Ratio of specific heats (C_p/C_v).
$\hat{k}$	Unit vector in the z-direction.
M	Mass. Note subscript designations.
$\hat{n}$	Unit vector. Outward normal with respect to the control volume, measured from the control volume surface.
p	Pressure. Note subscript designations.

s	Length of constant diameter section in rocket nozzle.
$S_{c.v.}$	Surface of control volume.
t	Time (Δt is a time increment).
u	Internal energy.
v	Volume. Note subscript designations.
V	Velocity (Fig. 1). Note subscript designations.
$\dot{W}_s$	Time rate of shear work done by the control volume on the surroundings.
z	Vertical coordinate.

Greek Symbols

ρ	Density.
τ	Shear stress.

Subscripts

a	Air condition.
b	Condition at a deformable bounding surface of the control volume.
c.v.	Condition with respect to the control volume.
e	Condition at the nozzle exit.
exterior	Condition at the exterior surface of the rocket.
H	Water surface condition.
interior	Condition at the rocket interior.
I	Initial condition.
o	Condition of the noninertial reference frame as seen from the inertial reference frame.
r	Relative to the noninertial reference frame affixed to the rocket
rn	Relative and normal to the surface of the control volume.
R	Rocket condition.
W	Water condition.

Superscripts

($\vec{\quad}$)	Vector quantity.
($\dot{\quad}$)	Time derivative.

REFERENCES

- Foss, J. F., "The Conservation Laws Via the Control Volume Approach, Some Laboratory Experiments," *Mechanical Engineering News*, ASEE, Vol. 5, No. 2, May 1968.
- Foss, J. F., and Heyer, D. E., "The Compressed Air-Water Rocket," available from Professor J. F. Foss, Dept. of Mechanical Eng., Michigan State Univ., East Lansing, Mich. 48823. A 16mm silent film from a high-speed data acquisition movie is also available on loan, or it may be purchased at cost (\$20).
- Hansen, A., *Fluid Mechanics*, John Wiley & Sons, New York, 1967.

Elementary Theory of Axially Loaded Helical Springs

MORRIS OJALVO
JOHN T. WILSON

The elementary theory of helical springs is used to obtain the rotation of any cross-section of the helix about its tangential axis and its displacement parallel to the axis of the spring. Without resorting to concepts too sophisticated for an elementary course in strength of materials, a greater and more accurate insight is provided than is available in elementary texts. For the condition of two loads coinciding with the axis of the spring and applied at the ends, it is shown that all of the twist of the helix is the result of displacement, and that the cross-sections do not rotate about the tangential axis of the helix.

The elementary theory of tightly wound helical springs subjected to axial force is approximate in that it assumes each coil of the helix to be stressed and to distort as though it were a ring lying in a plane perpendicular to the axis of the spring. The elementary theory also assumes that displacements and distortions are entirely due to a twisting of the helix about its tangential axis. Derivations based upon the elementary theory, while satisfactory in terms of their primary objectives, give at best scant insight into the way in which the helix distorts, and at worst present an inaccurate picture of the distortions (Den Hartog; Olsen; Shanley; Timoshenko and MacCullough). This shortcoming may be remedied without resorting to a sophisticated theory which may be unsuitable to a first course in strength of materials.

Consider the spring in Fig. 1. A point on the helix is located by the angle α . The twisting moment, H , at any section of the helix is PR . We have from Clebsch's assumptions (Love, p. 397)

$$H = PR = C(\tau_1 - \tau_0), \quad (1)$$

in which τ_1 is the final twist and τ_0 the initial twist of the helix. The torsional

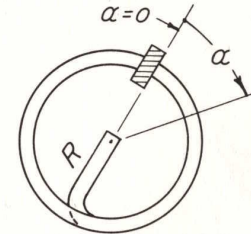


Fig. 1a. Helical spring.

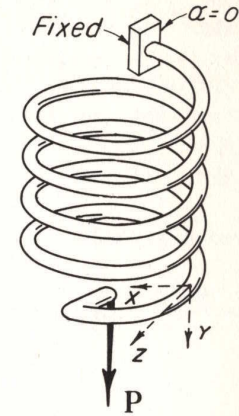


Fig. 1b. Helical spring coordinate system.

stiffness of the helix is C . If one of the principal centroidal axes of the unstressed helix is perpendicular to the binormal at each cross-section, we have, according to the elementary theory, $\tau_0 = 0$. According to the theory of small inextensible displacements due to St. Venant (Love, pp. 444-47),

$$\tau_1 = \frac{1}{R} \left(\frac{d\beta}{d\alpha} + \frac{1}{R} \frac{dv}{d\alpha} \right). \quad (2)$$

In Eq. 2, β is the rotation of any cross-section of the helix about its tangent, and v is the displacement of the helix parallel to the spring's axis. The positive sense of rotations, displacements, curvatures, and stress resultants follow the right-hand screw rule convention for the coordinate system shown in Fig. 1b. A derivation of the geometric relations given by Eq. 2 and 4, suitable to a first course in strength of materials, has been presented (Timoshenko, 1930, 1936). The moment at any section of the helix about the x -axis is designated G and is equal to zero. Again we have, according to Clebsch's assumptions (Love, p. 397),

$$G = A(\kappa_1 - \kappa_0) = 0, \quad (3)$$

in which A is the flexural stiffness about centroidal principal axis x , and κ_0 and κ_1 are initial and final curvatures of the helix about the x -axis. According to our elementary theory $\kappa_0 = 0$, and according to St. Venant's geometric relations (Love, pp. 444-47),

$$\kappa_1 = \frac{\beta}{R} - \frac{1}{R^2} \frac{d^2v}{d\alpha^2}. \quad (4)$$

Eq. 1 and 3 are the basic equations for the determination of β and v . After substituting from Eq. 2 and 4, these become

$$\frac{1}{R} \frac{dv}{da} + \frac{d\beta}{da} = \frac{PR^2}{C}, \quad (5)$$

and

$$\frac{1}{R} \frac{d^2v}{da^2} - \beta = 0. \quad (6)$$

Their solution yields

$$v = \frac{PR^3 a}{C} + C_1 \sin a + C_2 \cos a + C_3, \quad (7)$$

and

$$\beta = -\frac{1}{R} (C_1 \sin a + C_2 \cos a). \quad (8)$$

The terms C_1 , C_2 , and C_3 are constants of integration which can be determined from the boundary conditions at $a = 0$:

$$\beta = v = \frac{1}{R} \frac{dv}{da} = 0. \quad (9)$$

After evaluating the constants one obtains:

$$v = \frac{PR^3}{C} (a - \sin a), \quad (10)$$

and

$$\beta = \frac{PR^2}{C} \sin a. \quad (11)$$

The displacements v consist of a linear part and a part periodic in a which becomes insignificant when there are more than a few turns to the helix. In fact, the periodic part may be completely eliminated by giving the spring a small rigid body rotation of magnitude $-\frac{PR^2}{C}$ about the x -axis at $a = 0$. If this is done, the following is obtained:

$$v = \frac{PR^3 a}{C} \quad (12)$$

$$\beta = 0. \quad (13)$$

The latter condition could occur if, instead of fixing the helix at $a = 0$, a rigid radial arm extended to the center of the helix and if a reactive force were applied at this point along the axis of the spring as shown in Fig. 2. For this condition of support, points on the helix would displace uniformly, the cross-sections would not rotate about the z -axis, and the twist would be entirely due to vertical displacements v (Eq. 2).

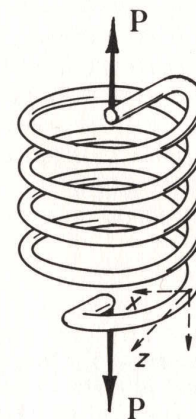


Fig. 2. Axially loaded helical spring.

The nonelementary theory of helical springs due to St. Venant and Kelvin is concerned with the deformations of a helix for which a principal centroidal axis of each cross-section coincides or makes a constant angle with a binormal of the helix both in the unstressed and in the stressed condition (Love, pp. 415-16). The foregoing implies the result given by Eq. 13. The result given by Eq. 12 is obtainable from the nonelementary theory by making those approximations which characterize the elementary theory, i.e., each coil of the helix lies in a plane which is perpendicular to the axis of the helix.

REFERENCES

- Den Hartog, J. P., *Strength of Materials*, McGraw-Hill, New York, 1949.
- Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, 4th Ed., Dover Publications, New York, 1944.
- Olsen, G. A., *Elements of Mechanics of Materials*, 2nd Ed., Prentice-Hall, Englewood Cliffs, N. J., 1966.
- Shanley, F. R., *Strength of Materials*, McGraw-Hill, New York, 1957.
- Timoshenko, S., *Strength of Materials, Part II*, Van Nostrand, New York, 1930, pp. 469-70.
- Timoshenko, S., *Theory of Elastic Stability*, 1st Ed., McGraw-Hill, New York, 1936.
- Timoshenko, S., and MacCullough, G. H., *Elements of Strength of Materials*, 3rd Ed., Van Nostrand, New York, 1949.

NOTE

Figures were drawn by Professor Fairfax E. Watkins of the Engineering Graphics Department at Ohio State University.

Growth of the Turbulent Flat-Plate Boundary Layer

JAMES M. ROBERTSON

A failing in the common presentations for the growth of a boundary layer on a smooth flat plate is noted and rectified. Instead of the layer growing with distance to the 4/5-power, as predicted by the old 1/7-power velocity profile relation, it is shown that up-to-date relations yield a variation more nearly represented by the 5/6-power, over a considerable Reynolds number range. Experimental results verifying the predictions are indicated.

The boundary layer along a smooth plane surface in the absence of a pressure gradient is commonly used to exemplify more general boundary layer occurrences. It is a useful reference in estimating boundary layer thickness and friction development about elongated bodies such as airfoils, ships, and streamlined trains, and for considering the entrance region flows of conduits.

In 1920, Prandtl and von Karman independently adapted Blasius' smooth-pipe friction-factor formulation to the turbulent flat-plate boundary layer flow problem. The indication was that the velocity in the layer should vary as the 1/7-power of the wall distance with the prediction that boundary layer thickness should then grow with the 4/5-power of the distance from the start or effective start of the turbulent layer.

Since this result was suggested, on the weak ground that boundary layer thickness corresponds to pipe radius, the turbulent flat-plate layer has been subjected to many studies. Several of these lead to more rational relations for the velocity-profile relation and expressions for the rate of boundary layer growth. Despite this advancement, all basic texts in fluid mechanics and most advanced texts and reference works continue to indicate the 1920 predictions as most appropriate. Often when an improvement is suggested, it is merely the 1925 suggestion of allowing the exponent to decrease as the flow Reynolds number of consideration becomes larger.

It is not hard to demonstrate that the 1/7-power velocity-profile relation is a poor fit to measured profiles near the wall and near the outer edge of the layer, even for the Reynolds number range for which it is claimed to be proper. Furthermore, the resultant integral relations and friction-factor predictions compare poorly with experimental results except at a single Reynolds number which differs for the several comparison bases. Nor has the growth-rate prediction really been subjected to experimental verification. Despite these demonstrable failings and lack of verification for the 1920 hypothesis, newer developments are rarely given attention. It is true that the contemporary relations for boundary layer growth are complex and generally only obtainable from tabulated functions. But these can be expressed as simple relations, to an adequate accuracy.

With the boundary layer (disturbance) thickness δ as the wall distance at which the velocity has reached to within 1% of the free stream value, the 1920 indication for the 1/7-power law velocity profile is that

$$\frac{\delta}{x} = \frac{0.37}{\text{IR}_x^{0.2}}, \quad (1)$$

and that the momentum thickness θ is given by

$$\frac{\theta}{x} = \frac{0.036}{\text{IR}_x^{0.2}}, \quad (2)$$

where x is the plate length to the point at which the δ or θ is found and $\text{IR}_x = u_1 x/\nu$. (Here u_1 = free stream velocity, ν = kinematic viscosity.) Two boundary layer shape factors are predicted as

$$G = \frac{\delta}{\theta} = 10.3 \quad \text{and} \quad H = \frac{\delta^*}{\theta} = 1.286, \quad (3)$$

where δ^* is the displacement thickness. Actually for flow past smooth boundaries, summaries of a large amount of experimental data indicate that H is not constant but decreases from a value of 1.6 at an IR_θ of 300 to 1.26 at 50,000; whereas G increases in this range from a value of over 9 to one of over 10 (Robertson, 1963; Ross, "Turbulent Flow," 1956).

The development of the logarithmic-type turbulent velocity-profile relation by von Karman and Prandtl in the early 1930's is indicated in many texts, although application to boundary layers is usually skimmed over lightly. It seems to have been missed that von Karman in 1934, by comparing expressions for the local skin-friction coefficient from pipe and flat-plate flows, also obtained a relation for the boundary layer growth as

$$\frac{\delta}{x} = 0.38 \sqrt{c_f}, \quad (4)$$

where c_f is the local skin-friction coefficient. This solution method assumes that δ of the flat plate corresponds to the pipe radius. A similar relation was derived (Robertson and Ross) on the slightly more rigorous basis that the viscous sublayer is the same for the two flows, as

$$\frac{\delta}{x} = 0.40 \sqrt{c_f}. \quad (4')$$

Relations for the two shape factors are then

$$G = \frac{1}{1.77 \sqrt{c_f} - 6.3 c_f}, \quad H = \frac{1}{1 - 3.5 \sqrt{c_f}} \quad (5)$$

Since c_f varies with IR_θ , these do imply a decrease in H and increase in G with IR_θ in fashion similar to that actually found. However the predictions are significantly off in magnitude (Robertson, 1963).

The reason for this failing of the universal-profile concept of the logarithmic law is that it is only applicable within wall distances of 10-15% of δ , as shown clearly in 1953 for the pipe case (Ross, 1953). For this wall-law region, the velocity profile outside the viscous sublayer was shown to be given by the expression

$$\frac{u}{u_\tau} = 5.6 + 5.6 \log \frac{yu_\tau}{\nu}, \quad (6)$$

where $u_\tau = \sqrt{\tau_w/\rho}$ is the shear velocity and y is the wall distance (Robertson, 1963; Ross: 1953; "A Physical Approach," 1956). This is termed the "law of the wall." For larger distances a "core law" of the three-halves-power-velocity-deficiency type applied. Due to the complexities of the profile functions, the usual shape factor was not found analytically, but the following function empirically represents the occurrences:

$$H = 1.09 + \frac{0.40}{0.72 \log IR_\theta - 1.0}. \quad (7)$$

This has been well verified through comparison with experimental data (Robertson, 1963). For the local skin friction coefficient Ross found

$$c_f = (4.4 + 3.8 \log IR_\theta)^{-2}. \quad (8)$$

About the same time, an extensive review of flat-plate boundary layer data was summarized mainly in tabular form (Coles). That c_f formulation is only slightly (3%) below that of Ross. For the away-from-the-wall region a formulation was developed in terms of a wake-like velocity function.

A significant indication by Coles concerning prediction of the flat-plate boundary layer growth seems to have been generally lost. For zero pressure gradient the boundary layer momentum-integral equation becomes simply

$$\frac{d\theta}{dx} = \frac{c_f}{2}, \quad (9)$$

whence

$$\frac{\theta}{x-x_0} = \frac{C_F}{2}, \quad (9')$$

where C_F is the average friction coefficient, defined as $\frac{1}{x-x_0} \int_{x_0}^x c_f dx$, and x_0 is a virtual origin of the turbulent layer. The virtual origin concept was first suggested by van der Hegge Zijnen in 1924 on the basis of Eq. 1. Following Prandtl's 1927 assumption, x_0 is often taken as zero, but this is not necessarily the proper assumption. Now Eq. 9' is the significant indication of Coles, i.e., that the boundary layer momentum thickness is simply found from the average friction coefficient C_F . The universal logarithmic-law velocity profile relation was integrated in the early 1930's to yield a c_f versus IR_x relation in the form $1/\sqrt{c_f}$ as function of IR_x c_f . A similar relation applies to C_F , and with constants found by Schoenherr in 1932, this is

$$\frac{1}{\sqrt{C_F}} = 4.13 \log (IR_{x-x_0} C_F).$$

As a direct relation for C_F , the Prandtl-Schlichting interpolation formula is usually employed instead:

$$C_F = \frac{0.455}{(\log IR_{x-x_0})^{2.58}}. \quad (10)$$

Alternatively, Coles' tabulated C_F values are available (Coles; Thwaites). The two sets of values do not differ greatly. When Eq. 10 is substituted into Eq. 9', the momentum thickness relation is found to be

$$\frac{\theta}{x-x_0} = \frac{0.228}{(\log IR_{x-x_0})^{2.58}}. \quad (11)$$

This prediction represents a considerable modification over the 1/7-power law result of Eq. 2.

As a closer approximation to the true occurrences, Eq. 9 has been solved (for $2 \times 10^4 \leq IR_x \leq 10^8$) for θ by direct numerical integration using c_f as given by Eq. 8. The result appears in tabular form, but can be approximated to within 1% over the IR_{x-x_0} range of 5×10^4 to 5×10^7 by the expression

$$\frac{\theta}{x-x_0} = \frac{0.161}{(\log IR_x)^{2.42}} \quad (12)$$

In 1956 the boundary layer momentum equation was integrated to obtain a relation for the growth of θ in the pipe entrance region (Ross, "Turbulent Flow," 1956). In the flat-plate limit this reduces to the result

$$\frac{\theta}{x-x_0} = \frac{0.0215}{IR_{x-x_0}^{1/6}} \quad (13)$$

A similar relation, with the constant about 2% less, fits the results of the numerical integrations to within 1% but over a restricted range $5 \times 10^5 \leq IR_x \leq 10^7$. Now turbulent layers may well be of interest at Reynolds numbers outside this latter range but are not so likely to be found outside the broader range limitation of Eq. 12.

When δ is desired, these θ relations suffer from the difficulty that the shape factor $G = \delta/\theta$ is not well established. The difficulty regarding the shape factor $G = \delta/\theta$ should be surmountable via integration of appropriate velocity-profile relations. However, those which are available do not seem to agree well with experiments in predictions of G . Thus G has to be found by reference to empirical correlations. Although the needed data are not plentiful (one major difficulty is that δ is not precisely definable), the general indications are that G has a value of about 10 and increases slowly with IR_θ on a logarithmic scale. Some information on G has been compiled (Robertson, 1963); with additional data collected since then, a simple tentative correlation for smooth surfaces is

$$G = 8.22 + 0.475 \log IR_\theta \quad (14)$$

The prediction of Eq. 3 is reasonably close to this; whereas that of Eq. 5 is high and that of Coles below this.

On the basis of Eq. 14 for G , the $\theta/(x-x_0)$ results found by integration have been modified to yield the desired variation in $\delta/(x-x_0)$ with IR_{x-x_0} . This result is presented in Fig. 1 in comparison with the predictions of Eq. 1 and 4'. It is seen that the 1/7-power-law result of Eq. 1 is only appropriate at a Reynolds number of about 10^7 , whereas the result from the universal log-law (Eq. 4') diverges markedly except at about 10^5 . For comparison purposes Fig. 1 also includes the laminar boundary layer result from Blasius' solution.

The relative boundary layer disturbance thickness δ/x decreases rapidly with the effective Reynolds number IR_{x-x_0} . The variation is about as the minus 1/6-

power, and, for the range $10^5 < IR_{x-x_0} < 10^7$, the following simple expression is found to be reasonably accurate (within 2%):

$$\frac{\delta}{x-x_0} = \frac{0.21}{IR_{x-x_0}^{1/6}} \quad (15)$$

The form of this result is one of convenience rather than the outcome of any power-law formulation. The agreement of this expression with the exact integral result is shown in Fig. 1; outside the limits of applicability it is seen to underindicate the boundary layer thickness.

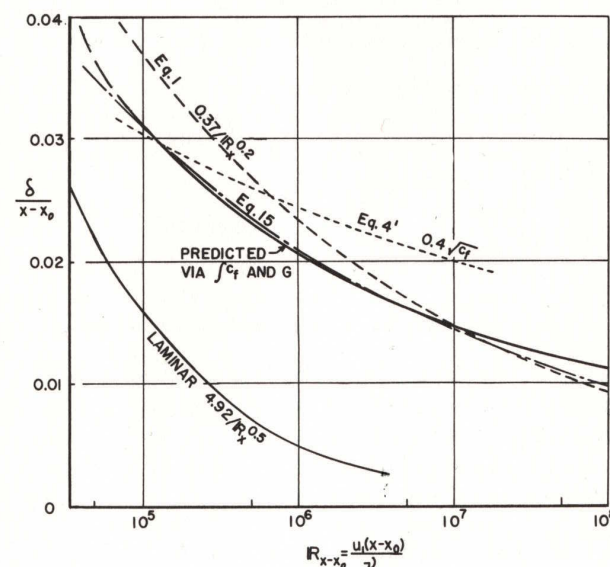


Fig. 1. Boundary layer growth rate predictions.

The desirability of direct experimental verification of boundary layer growth predictions has been noted. Such verification is offered in Fig. 2 as IR_δ versus IR_x from turbulent flat-plate boundary layer studies (Peters; Schultz-Grunow; Klebanoff and Diehl; Landweber and Siao; Smith and Walker) at several Mach numbers (IM) in the low subsonic range. Generally good agreement with the prediction is apparent, although some divergence in the Landweber and Siao data appears at the highest Reynolds numbers. Comparison with the θ predictions is also quite good except for the data of Schultz-Grunow at high Reynolds numbers. The appearance of significant virtual origin distances x_0 in some of the studies is apparent. With special trip devices (as sand roughness near leading edge) large negative values of x_0 are even found. For flat plates with rounded leading edges, x_0 is usually about zero, as first shown by Hansen (Hansen).

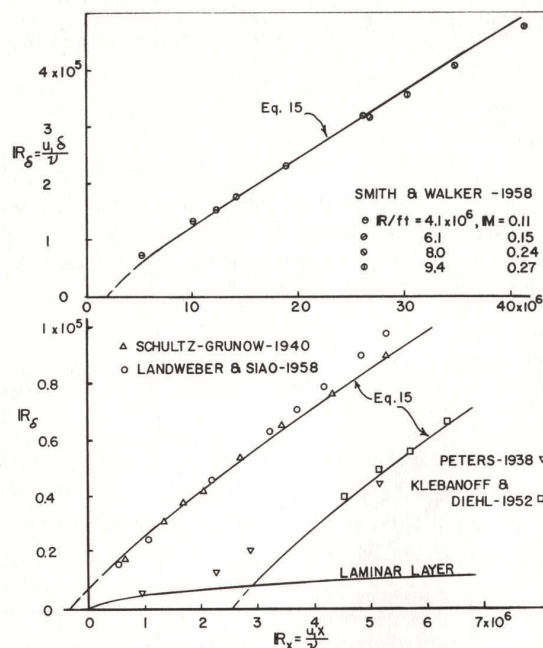


Fig. 2. Verification of boundary layer growth relation.

The failing of the ubiquitous $1/7$ -power-law prediction for turbulent boundary layer growth has thus been fully demonstrated, and the results of a more proper solution shown to be expressible in an equally simple form. Thus, for the momentum thickness Eq. 12 gives a precise indication of the growth relation. The more approximate relations of Eq. 13, as well as Eq. 15 for δ , involving $IR_{x-x_0}^{1/6}$, are quite tractable. Their form is not the result of any power-law type profile relation, however. In fact, the velocity profile formulation must involve a division into at least two regions, one near the wall involving the viscous sublayer and the turbulent law of the wall and the other a core law at larger distances from the wall.

REFERENCES

- Coles, D., "The Problem of the Turbulent Boundary Layer," *Journal of Applied Mathematics and Physics (Zeitschrift für angewandte Mathematik und Physik)*, Vol. 5, 1954, pp. 181-203.
- Hansen, M., "The Velocity Distribution in the Boundary Layer on a Submerged Plate," *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 8, 1928, pp. 185-199, translated from German in NACA Tech. Memo, 585, 1930.
- Klebanoff, P. S., and Diehl, Z. W., "Some Features of Artificially Thickened Fully Developed Turbulent Boundary Layers with Zero Pressure Gradient," NACA Report 1110, 1952.
- Landweber, L., and Siao, T. T., "Comparison of Two Analyses of Boundary-Layer Data on a Flat Plate," *Journal of Ship Research*, March 1958, pp. 21-33.

- Peters, H., "A Study in Boundary Layers," *Proceedings, Fifth International Congress in Applied Mechanics*, 1938, pp. 393-395.
- Robertson, J. M., "A Turbulence Primer," University of Illinois Engineering Experiment Station, Circular 79, 1963.
- Robertson, J. M., and Ross, D., "Water Tunnel Working Section Flow Studies," Ordnance Research Laboratory, The Pennsylvania State University, Report 7958-97, June 1948.
- Ross, D., "A New Analysis of Nikuradse's Experiments on Turbulent Flow in Smooth Pipes," *Proceedings, Third Midwest Conference on Fluid Mechanics*, University of Minnesota, 1953, pp. 651-667.
- Ross, D., "A Physical Approach to Turbulent Boundary-Layer Problems," *Transactions, American Society of Civil Engineers*, Vol. 121, 1956, pp. 1219-1254.
- Ross, D., "Turbulent Flow in the Entrance Region of a Pipe," *Transactions, American Society of Mechanical Engineers*, Vol. 78, 1956, pp. 915-923.
- Schultz-Grunow, F., "New Frictional Resistance Law for Smooth Surfaces," *Luftfahrtforschung*, Vol. 17, 1940, pp. 239-246; translated from German in NACA Tech. Memo, 986, 1941.
- Smith, D. W., and Walker, J. H., "Skin Friction Measurements in Incompressible Flow," NACA Tech. Note 4231, 1958.
- Thwaites, B., Ed., *Incompressible Aerodynamics*, Oxford University Press, 1960, p. 73.
- Von Karman, T., "Turbulence and Skin Friction," *Journal of Aeronautical Sciences*, Vol. 1, 1934, pp. 1-20.

Experimental Mechanics for Undergraduates

BRUCE G. ROGERS

A three-hour course in experimental and theoretical mechanics is taught to third-year engineering students at Lamar State College of Technology. The course combines theory with laboratory work, emphasizing the use of experimental models to illustrate theoretical concepts. Students seem to have a better grasp of structural behavior after taking the course in this format.

Elementary mechanics is taught to most engineering undergraduates in a sequence of courses beginning with statics and dynamics of particles and rigid bodies. The next course usually deals with the mechanics of deformable bodies, often called strength of materials. After this course, the sequence varies according to the type of engineering. For example, mechanical engineering students may begin a sequence in machine design, while civil engineering students begin courses in structural analysis. Because such courses are based upon fundamental principles of mechanics, and because the usual three semesters of mechanics are frequently not sufficient to provide students with adequate background in this subject, a considerable portion of junior-level courses in structures or machines must be devoted to further instruction in mechanics.

One alternative to this situation is the addition of another semester in mechanics. However, the engineering undergraduate curriculum is so crowded that a new course can hardly be justified unless credit hours are reduced in other courses. Therefore, a more advanced course in mechanics should be closely related to subsequent design courses. Such a relationship is readily developed if laboratory work is incorporated into the theoretical course. A comprehensive survey of laboratory instruction for engineering students has been completed recently (ASEE). Several features of laboratory instruction have been combined in an experimental mechanics course at Lamar State College of Technology.

DEVELOPMENT OF NEW COURSE

A three semester-hour course in experimental and theoretical mechanics has been taught to third-year undergraduate engineering students at Lamar Tech for the past three years. The first two years were primarily used to develop appropriate laboratory demonstrations. A grant from the National Science Foundation, under the instructional scientific equipment program, provided additional funds for the purchase of laboratory equipment. The course has now been taught twice with the newly equipped laboratory. Although modifications will probably be made, the basic pattern of the course seems to be established.

The textbook used is *Experimental Stress Analysis* (Dally and Riley). Part I of the text introduces the student to elementary elasticity theory, and the remaining parts describe brittle coatings, photoelasticity, and strain measurements and instrumentation. Because many different experimental techniques are used, the average student does not have time to become proficient in the use of most of the apparatus, and experiments are usually conducted or closely supervised by the instructor. However, each student is assigned a role and must assume some responsibility for part of each experiment. He is thus encouraged to exercise reasonable care, although he is not expected to become very skillful in laboratory technique. In fact, the purpose of the laboratory portion of the course is explained repeatedly: to use experimental methods to illustrate theoretical concepts. Primary emphasis is placed on theoretical stress analysis.

EXPERIMENTS USED

Experiments are conducted in weekly laboratory periods which are three hours long. The first four experiments use brittle coatings and photoelasticity to portray stress concentration effects. In the final experiment of this series, photographs are taken with a camera with Polaroid back, and attempts are made to measure the value of the stress concentration factor in a simple model. At this stage the student has had enough theory of elasticity to understand how simple stress concentration factors are calculated analytically.

Instead of a separate laboratory report on each experiment, the student is required to write a comprehensive report on the entire subject of stress concentrations, incorporating his observations and his interpretation of them. He is also to use library resources to obtain additional information on stress concentrations and their significance. Considerable attention is given to the student's own conclusions, based on his library research, analytical calculations, and observations of the laboratory demonstrations.

One item used in stress concentration experiments is a transmitted light polariscope (Fig. 1), constructed by Lamar Tech laboratory technicians from plans developed at the University of Notre Dame. In addition, a standard reflected light polariscope and equipment for applying brittle coatings are used.

The next six experiments deal with topics ordinarily considered separately, grouped together in this course because essentially the same apparatus is used for each experiment. The topics are pure bending, bending with shear,

unsymmetrical bending, torsion of circular sections, and torsion of noncircular sections. By now, the student is familiar with the limitations of the classical strength of materials formulas for bending and torsion because he has been given a theory of elasticity approach.

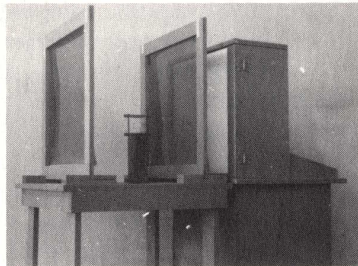


Fig. 1. Diffused light polariscope.

The first two experiments of the series consist in making strain measurements with electrical resistance strain gages mounted on small models loaded with dead weights. The remaining experiments use a large steel beam (8 WF28) instrumented with strain gages and loaded with a hand-pumped hydraulic ram system (Fig. 2, 3, 4). Loads are measured with a commercial electronic load indicator which uses load cells.

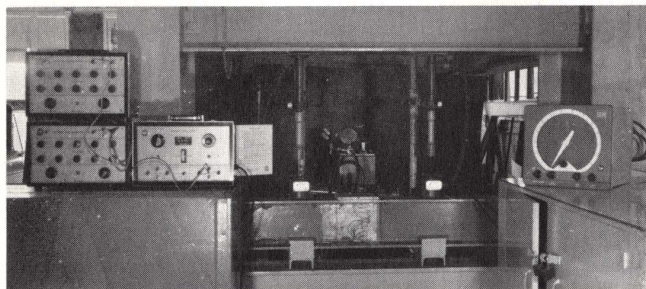


Fig. 2. Instrumented test beam in loading frame. Strain reading equipment at left, test beam in center, and load indicator at right.

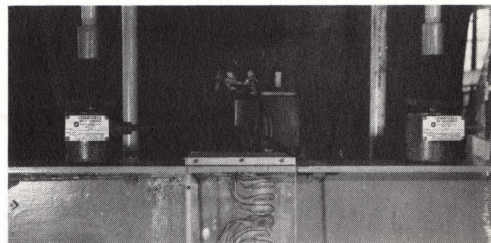


Fig. 3. Close-up of test beam showing load cells and hydraulic rams in position. Hand-operated pump is in background.

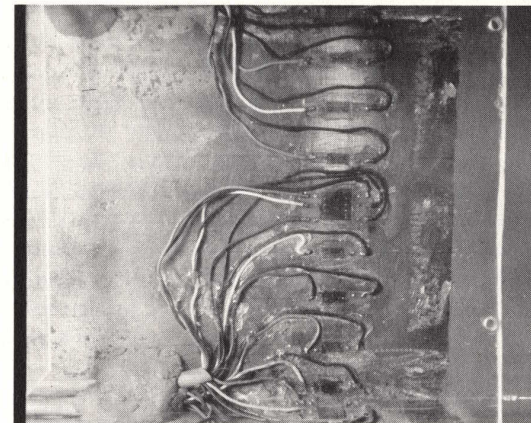


Fig. 4. Strain gages mounted on test beam. Note rosette gage on neutral axis.

Results confirming the strength of materials formulas for pure bending and circular torsion are usually obtained, and the student acquires confidence in the measuring apparatus. However, a small departure from ideal loading conditions may cause large differences in strains, and these departures are deliberately introduced in some experiments. By applying loads in various locations as well as changing the supports, a number of different loading conditions may be obtained. Thus, experiments may be changed each time the course is taught without basic modifications of the apparatus.

The final series of experiments is concerned with dynamic loading. A strain gage oscilloscope and a stroboscope demonstrate both impact loads and free vibrations. These experiments are designed to present the student with qualitative rather than precise quantitative information, and to stimulate discussion of the importance of dynamic loadings. Library resources are used to provide examples of wind-induced vibrations, foundation problems caused by the operation of heavy machinery, pile driving, and similar topics.

RESULTS

The value of the course has been demonstrated in the senior structural design courses, where the student seems to have a better grasp of structural behavior than his counterpart who took a strictly theoretical course in advanced strength of materials. There was a laboratory course in materials testing in the previous curriculum, but it was not well correlated with the theoretical course. By combining both theory and laboratory work, a more effective approach has been achieved.

REFERENCES

- American Society for Engineering Education, *Journal of Engineering Education*, articles on engineering laboratory teaching, Vol. 58, No. 3, Nov. 1967.
- Dally, J. W., and Riley, W. F., *Experimental Stress Analysis*, McGraw-Hill, New York, 1965.

St. Venant's Problem in Cylindrical Coordinates

ROBERT SCHMIDT

Four different formulations of an elastic cantilever beam problem are formulated as boundary value problems. This makes it easier for the student to understand St. Venant's problem of bars in a course on theory of elasticity. Two examples illustrate the application of the derived equations.

The equilibrium of an elastic prismatic beam subjected to loads at its ends is known as St. Venant's problem of bars. In this paper the name is used to identify the problem of flexure of an elastic cantilever beam subjected to a transverse load—parallel to one of the principal axes of the normal cross-section—which is distributed in a certain fashion at the free end of the beam. This is one of the most general cases of problems concerned with prisms that have been simplified enough to permit a series of analyses of some practical importance. Originally, the problem under consideration was formulated in rectangular coordinates by means of the semi-inverse method by St. Venant, who obtained a two-dimensional Laplace differential equation and a complicated boundary condition on the lateral surface of the long prismatic beam (Love). Much later, and again in rectangular coordinates, Timoshenko reformulated the problem in a very elegant way by making use of a stress function and an additional arbitrary function of one variable only, and obtained a two-dimensional differential equation with a simple boundary condition (Timoshenko and Goodier).

This paper, following the main idea of Timoshenko, establishes four different formulations of the cantilever problem in cylindrical coordinates. These formulations make it easier for the instructor to present known results of the problems solved previously, as well as to facilitate the application of numerical methods to the solution of practical problems.

ST. VENANT'S PROBLEM

27

FORMULATIONS OF THE PROBLEM

In order to fix ideas, a rectangular coordinate system x, y, z is taken in such a way that the x -axis is vertical and points downward, the z -axis is parallel to the generators of the prism (cylinder), and the y -axis completes the right-handed system of coordinates. The origin of this coordinate system is situated in the plane coincident with the xy plane of the rear end of the cantilever beam, which is of length L ; the rear end plane is the one seen by looking in the direction of the positive z -axis. The z -axis of the cylindrical coordinate system is coincident with the z -axis of the rectangular coordinate system; the angle θ is measured from the negative x -axis toward the positive y -axis and indicates the orientation of the radial line segment r which emanates from the origin. The two coordinate systems are related by the equations

$$x = -r \cos \theta, \quad y = r \sin \theta, \quad (1)$$

or

$$x^2 + y^2 = r^2, \quad \tan \theta = -y/x. \quad (2)$$

According to St. Venant (Timoshenko and Goodier),

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \quad \sigma_z = Cz(R + x) \quad (3)$$

for the present problem. The notations σ and τ for stresses require no explanation. C is a constant depending on the vertical force P applied to the rear end of the beam and on the shape of the normal cross-section of the cantilever beam. R is the distance measured along the x -axis from the z -axis to the centroid of the cross-section of the beam. In the present cylindrical coordinates, the assumptions of St. Venant (Eq. 3) take on the following form:

$$\sigma_r = \sigma_\theta = \tau_{r\theta} = 0, \quad \sigma_z = Cz(R - r \cos \theta). \quad (4)$$

With these assumptions, and in the absence of body forces, the equilibrium equations in cylindrical coordinates (Love) reduce to

$$\tau_{rz,z} = 0 \quad (5a)$$

$$\tau_{\theta z,z} = 0 \quad (5b)$$

$$\tau_{rz,r} + \frac{1}{r} \tau_{\theta z,\theta} + C(R - r \cos \theta) + \frac{1}{r} \tau_{rz} = 0, \quad (5c)$$

in which a comma before a subscript indicates partial derivative with respect to that subscript. From the first two of these equations it follows that

$$\tau_{rz} = \tau_r(r, \theta), \quad \tau_{\theta z} = \tau_\theta(r, \theta). \quad (6)$$

The third equilibrium equation (Eq. 5c) can be satisfied by letting

$$r\tau_r = \phi_{,\theta}, \quad \tau_\theta = -\phi_{,r} - Cr(R\theta - r \sin \theta) + f(r), \quad (7)$$

or

$$\tau_\theta = \psi_{,r}, \quad \tau_r = -\frac{1}{r} \psi_{,\theta} - Cr(\frac{1}{2}R - \frac{r}{3} \cos \theta) + g(\theta), \quad (8)$$

where $f(r)$ and $g(\theta)$ are arbitrary functions of integration, and $\phi(r, \theta)$ and $\psi(r, \theta)$ are usually referred to as stress functions, to be determined from the compatibility and boundary conditions.

The six compatibility conditions (Timoshenko and Goodier) reduce to two simple equations for the present problem:

$$\frac{\partial}{\partial x} (\tau_{yz,x} - \tau_{xz,y}) = 0 \quad (9a)$$

$$\frac{\partial}{\partial y} (\tau_{yz,x} - \tau_{xz,y}) = -\frac{2\nu GC}{E} \quad (9b)$$

in view of Eq. 1, 3, and 6, and the Hooke law (Timoshenko and Goodier),

$$\begin{aligned} E\epsilon_x &= -\nu\sigma_z, & E\epsilon_y &= -\nu\sigma_z, & E\epsilon_z &= \sigma_z, \\ \gamma_{xy} &= 0, & G\gamma_{yz} &= \tau_{yz}, & G\gamma_{xz} &= \tau_{xz}, \end{aligned} \quad (10)$$

in which ϵ 's and γ 's are very common notations for extensional and shearing strains, ν is the Poisson ratio, E is the modulus of elasticity (Young's modulus), and

$$G = E/2(1 + \nu) \quad (11)$$

is the modulus of rigidity or the modulus of elasticity in shear.

At this point it becomes convenient to introduce the auxiliary notation

$$\eta = \tau_{yz,x} - \tau_{xz,y}. \quad (12)$$

But

$$\frac{\partial}{\partial x} = -\cos \theta \frac{\partial}{\partial r} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \quad (13a)$$

$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}, \quad (13b)$$

and

$$\tau_{xz} = \tau_\theta \sin \theta - \tau_r \cos \theta \quad (14a)$$

$$\tau_{yz} = \tau_\theta \cos \theta + \tau_r \sin \theta, \quad (14b)$$

so that

$$\eta = \frac{1}{r} \left[\tau_{r,\theta} - (r\tau_\theta)_{,r} \right]. \quad (15)$$

Now Eq. 9 may be rewritten in the form:

$$\eta_{,x} = 0, \quad \eta_{,y} = -2\nu CG/E,$$

or

$$-\cos \theta \eta_{,r} + \frac{\sin \theta}{r} \eta_{,\theta} = 0$$

$$\sin \theta \eta_{,r} + \frac{\cos \theta}{r} \eta_{,\theta} = -2\nu CG/E,$$

from which

$$\eta_{,r} = -2\nu CG \sin \theta / E \quad (16a)$$

$$\eta_{,\theta} = -2\nu CGr \cos \theta / E. \quad (16b)$$

Integration of these equations yields

$$\eta = -2\nu CGr \sin \theta / E + F(r) \quad (17a)$$

$$\eta = -2\nu CGr \sin \theta / E + H(\theta). \quad (17b)$$

Hence

$$F = H = k = \text{constant}.$$

Thus, there remains only one compatibility equation,

$$\eta = -2\nu CGr \sin \theta / E + k,$$

or, in view of Eq. 15,

$$(r\tau_\theta)_{,r} - \tau_{r,\theta} = 2\nu CGr^2 \sin \theta / E - kr, \quad (18)$$

which may be expressed in terms of the stress functions ϕ and ψ .

Thus

$$\nabla^2 \phi = \frac{1}{r} \frac{d}{dr} \left[r f(r) \right] - 2RC \theta + (3 + 2\nu) Cr \sin \theta / (1 + \nu) + k, \quad (19)$$

or

$$\nabla^2 \psi = \frac{1}{r^2} \frac{d}{d\theta} g(\theta) - (1 - 2\nu) Cr \sin \theta / 3(1 + \nu) - k, \quad (20)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (21)$$

is the plane Laplacian operator.

The constant k is zero, if the beam bends without twisting.

A somewhat different approach can be taken in formulating the problem under consideration. Instead of making use of the stress functions ϕ or ψ , a different kind of function may be introduced that will satisfy the compatibility equation (Eq. 18) instead of the equilibrium equations. This can be done by defining the nonvanishing shearing stress components in the following two ways:

$$\tau_{\theta} = \frac{1}{r} \omega_{,\theta}, \quad \tau_r = \omega_{,r} + 2\nu CGr^2 \cos \theta / E + kr\theta + i(r), \quad (22a)$$

or

$$\tau_r = \chi_{,r}, \quad \tau_{\theta} = \frac{1}{r} \chi_{,\theta} + 2\nu CGr^2 \sin \theta / E - \frac{1}{2} kr + \frac{1}{r} j(\theta). \quad (22b)$$

Substitution of these expressions in the equilibrium equation (Eq. 5c) yields

$$\nabla^2 \omega = -i'(r) - \frac{1}{r} i(r) - CR + (1 - 2\nu) Cr \cos \theta / (1 + \nu) - 2k\theta, \quad (23)$$

or

$$\nabla^2 \chi = -\frac{1}{r^2} j'(\theta) - CR + (3 + 2\nu) Cr \cos \theta / 3(1 + \nu), \quad (24)$$

wherein the primes indicate the derivatives with respect to the variable appearing in the parentheses.

Thus, four forms (Eq. 19, 20, 23, and 24) of the governing equation of the problem under consideration have been established. The particular solutions due to the right sides of the governing equations are not difficult to find by inspection. They are

$$\phi_p = \int f(r) dr - \frac{1}{2} RCr^2 \theta + (3 + 2\nu) Cr^3 \sin \theta / 8(1 + \nu) + \frac{1}{4} kr^2 \quad (25)$$

$$\psi_p = \int g(\theta) d\theta - (1 - 2\nu) Cr^3 \sin \theta / 24(1 + \nu) - \frac{1}{4} kr^2 \quad (26)$$

$$\omega_p = -\int i(r) dr - \frac{1}{4} CCr^2 + (1 - 2\nu) Cr^3 \cos \theta / 8(1 + \nu) - \frac{1}{2} kr^2 \theta \quad (27)$$

$$\chi_p = -\int j(\theta) d\theta - \frac{1}{4} CCr^2 + (3 + 2\nu) Cr^3 \cos \theta / 24(1 + \nu). \quad (28)$$

These functions satisfy the differential equations, Eq. 19, 20, 23, and 24.

A set of functions satisfying the Laplacian equation, say

$$\nabla^2 \phi_h = 0, \quad (29)$$

may be obtained by the method of separation of variables and by trial and error. In this manner,

$$\begin{aligned} \phi_h = & \sum_n (A_n r^n + B_n r^{-n}) \cos n\theta + \sum_n (C_n r^n + D_n r^{-n}) \sin n\theta \\ & + K_1 \theta \log r + K_2 \log r + K_3 \theta + K_4 + \dots, \end{aligned} \quad (30)$$

and

$$\phi = \phi_p + \phi_h; \quad (31)$$

wherein A_n , B_n , C_n , D_n , K_1 , K_2 , K_3 , K_4 , and n are, at this stage, arbitrary constants.

The next step is to consider the boundary conditions. The lateral surface of the beam must be free from tractions; therefore (Timoshenko and Goodier)

$$\tau_{xz} \cos(x, N) + \tau_{yz} \cos(y, N) = 0, \quad (32)$$

where $\cos(x, N)$, $\cos(y, N)$, 0 are the direction cosines of the outward normal to the lateral surface of the cantilever beam. Eq. 32 may be rewritten in the forms:

$$\tau_{xz} \frac{dy}{ds} - \tau_{yz} \frac{dx}{ds} = 0, \quad (33)$$

or

$$\tau_{xz} \frac{dx}{dN} + \tau_{yz} \frac{dy}{dN} = 0, \quad (34)$$

wherein ds is the elementary arc length on the boundary line of a normal cross-section of the beam and N is the outward normal to the lateral surface. After some elementary transformations, the foregoing boundary conditions may assume the forms:

$$\tau_{\theta} \frac{dr}{ds} - r \tau_r \frac{d\theta}{ds} = 0, \quad (35)$$

or

$$\tau_r \frac{dr}{dN} + r \tau_{\theta} \frac{d\theta}{dN} = 0, \quad (36)$$

from which

$$\frac{d\phi}{ds} = \left[f(r) - C(Rr\theta - r^2 \sin \theta) \right] \frac{dr}{ds} \quad (37)$$

$$\frac{d\psi}{ds} = \left[g(\theta) - Cr^2 \left(\frac{1}{2}R - \frac{r}{3} \cos \theta \right) \right] \frac{d\theta}{ds} \quad (38)$$

$$\frac{d\omega}{dN} = - \left[i(r) + kr\theta + vCr^2 \cos \theta / (1 + \nu) \right] \frac{dr}{dN} \quad (39)$$

$$\frac{dX}{dN} = - \left[j(\theta) - \frac{1}{2}kr^2 + vCr^3 \sin \theta / 3(1 + \nu) \right] \frac{d\theta}{dN} \quad (40)$$

It should be noted at this point that on the lateral boundary of the cantilever beam, if neither r nor θ are constant, r and θ are not independent of each other, so that the expressions in the brackets in the boundary conditions, Eq. 37-40, are either functions of r or θ only, or constant, or zero. Also, along the lines $r = \text{const.}$ $dr/ds = 0$ and along $\theta = \text{const.}$ $d\theta/ds = 0$, so that the right sides of Eq. 37-40 vanish under certain circumstances and the boundary conditions simplify greatly. Thus, solutions of given problems are noticeably simplified if, in general, the arbitrary functions f , g , i , or j are chosen in such a way as to make the right sides of the boundary conditions vanish.

Thus, the problem of the cantilever is formulated as any one of the four boundary value problems, namely: Eq. 19 and 37, or Eq. 20 and 38, or Eq. 23 and 39, or Eq. 24 and 40.

EXAMPLE 1

As the first illustration of the application of the derived equations, consider the well-known problem of the solid cantilevered beam with circular cross-section of radius b . The best formulation of this problem is in terms of the function ϕ , Eq. 19 and 37. Since $r = b$ on the boundary, $dr/ds = 0$ in Eq. 37, and $f(r)$ can be assumed to vanish. With $R = 0$ and $k = 0$, Eq. 19 and 37 become

$$\nabla^2 \phi = (3 + 2\nu)Cr \sin \theta / (1 + \nu) \quad (41)$$

$$\phi = \text{constant} = 0 \text{ on } r = b. \quad (42)$$

From Eq. 25 and 30,

$$\phi = \phi_p + \phi_h = (3 + 2\nu)C(r^3 - b^2r) \sin \theta / 8(1 + \nu), \quad (43)$$

where $C = -P/I$. The form of ϕ_h is determined by the form of ϕ_p , which is known. The arbitrary constant C_1 is calculated from the simple boundary condition, Eq. 42. No guessing of solutions has been necessary. The shearing stress components can be easily calculated by differentiating Eq. 43 as indicated by Eq. 7.

EXAMPLE 2

Consider a circular annulus with a slit of infinitesimal width at the bottom, i.e., at $\theta = \pi$. Let the internal and external radii be denoted by a and b , respectively. In this case, it is natural to let R vanish, and $k = 0$ for torsionless bending. On $r = a, b$, $dr/ds = 0$; on $\theta = \pm\pi$, $\sin \theta = 0$. Hence, if $f(r) = 0$ in Eq. 37, $d\phi/ds = 0$ or, simpler, $\phi = 0$ on the boundary of the slit annulus.

With the foregoing results, Eq. 25 yields

$$\phi_p = (3 + 2\nu)Cr^3 \sin \theta / 8(1 + \nu). \quad (44)$$

The form of Eq. 44 suggests

$$\phi_h = (C_1r + D_1/r) \sin \theta \quad (45)$$

from Eq. 30, so that

$$\phi = [C_1r + D_1/r + (3 + 2\nu)Cr^3 / 8(1 + \nu)] \sin \theta. \quad (46)$$

The function ϕ satisfies the boundary condition along the slit, since $\sin(\pm\pi) = 0$. The constants C_1 and D_1 are calculated from the condition $\phi = 0$ on $r = a, b$. Thus

$$C_1 = -(3 + 2\nu)(a^2 + b^2)C / 8(1 + \nu) \quad (47)$$

$$D_1 = (3 + 2\nu)a^2b^2C / 8(1 + \nu), \quad (48)$$

and

$$\phi = [(3 + 2\nu)C / 8(1 + \nu)] [r^3 - (a^2 + b^2)r + a^2b^2/r] \sin \theta \quad (49)$$

in which $C = -P/I$. This ϕ is the same as for the annular cross-section.

This example should illustrate the usefulness of the foregoing formulations of the St. Venant problem. More difficult examples would make this article a technical paper, which was not the intention of the author. The purpose of this article is to help the instructor with selection of problems that can be assigned to students in a course on the theory of elasticity.

REFERENCES

- Love, A. E. H., *A Treatise on Mathematical Theory of Elasticity*, 4th Ed., Dover Publications, New York, pp. 331-332.
- Timoshenko, S., and Goodier, J. N., *Theory of Elasticity*, 2nd Ed., McGraw-Hill, New York, 1951, p. 319.